Basis Functions

It is often useful to approximate or represent a function by a simpler function. The most immediate functions that come to mind for this purpose is the use of piecewise polynomial[1] functions with polynomials of low degree. There are other important classes of approximation such as the representing periodic functions by a Fourier series[2]. In this document we consider the *basis functions* which are the simple functions which act as building blocks for the function that is to be represented.

In general let χ_1, χ_2 ,..., χ_n be the basis functions. The approximation to the function *f(x)* is written as follows:

$$f(x) \approx \sum_{i=1}^{n} \alpha_i \, \chi_i(x),$$

where the α_i are constants.

Basis functions are used extensively in the development of computational methods. For example the Finite Element Method[3] and the Boundary Element Method[4] are derived through the use of basis functions.

Polynomial Basis Functions

For polynomial basis functions, the function $f(x)$ and its approximation are 'matched' at a set of points $x_1, x_2, \ldots, x_n$ and the basis functions χ_1, χ_2 ,..., χ_n are usually chosen so that

$$\chi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

and in this case

$$f(x) \approx \sum_{i=1}^{n} f_i \, \chi_i(x),$$

where $f_i = f(x_i)$.

The following data will be used to illustrate each form of polynomial basis function.

x_i	0	1	2	3	4	5	6	7	8
$f_i = f(x_i)$	0	2	1	4	5	4	5	4	7

The basis functions do not have to be uniform in the sense that they may have different widths. However, for simplicity, in this document we will use uniform basis functions for the purposes of illustration.

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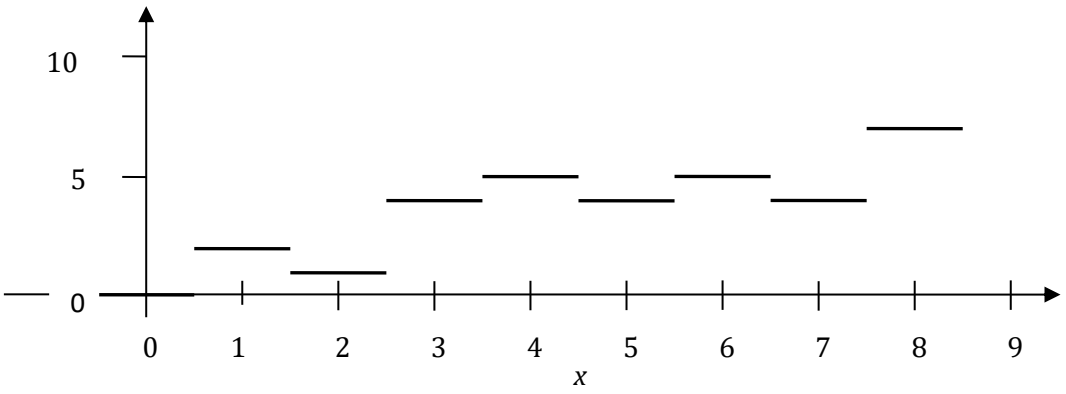
[1] Piecewise Polynomial Interpolation
[2] Fourier Series
[3] Finite Element Method
[4] Boundary Element Method

The approximation to the function (x) , as shown in the graph at the top of the page can be written as follows:

$$f(x) \approx 0\chi_1(x) + 2\chi_2(x) + 1\chi_3(x) + 4\chi_4(x) + 5\chi_5(x) + 4\chi_6(x) + 5\chi_7(x) + 4\chi_8(x) + 7\chi_9(x)\,.$$

Constant basis functions

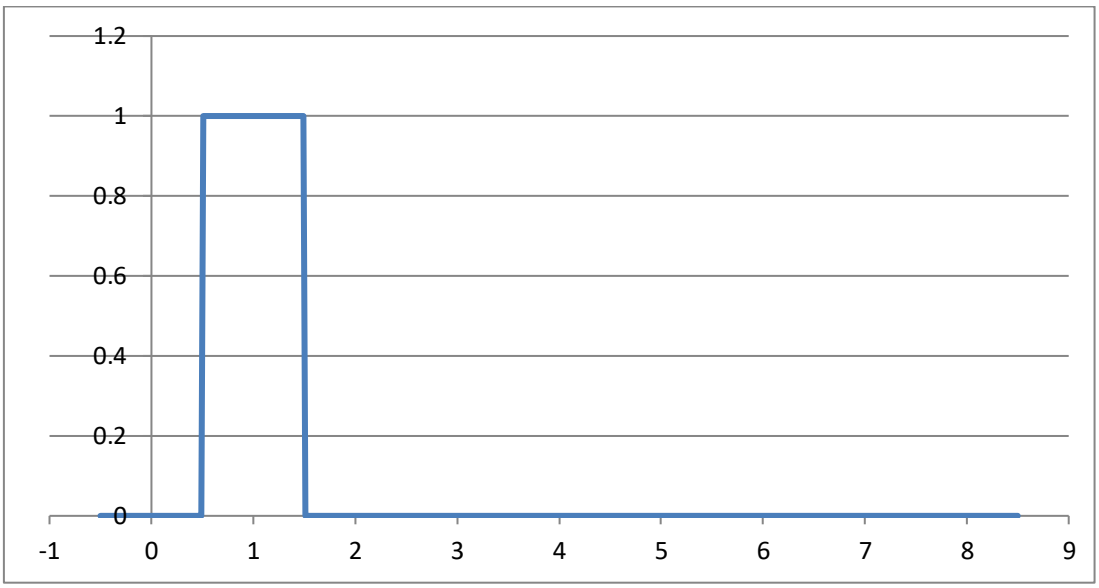
The representation of the data using a piecewise constant approximation is shown in the following graph.



For this approximation, the basis functions are defined as follows:

$$\chi_i(x) = \begin{cases} 1, & i - \frac{3}{2} \le x < i + \frac{3}{2} \\ 0, & x > i - \frac{3}{2} \text{ or } x \ge i + \frac{3}{2} \end{cases}.$$

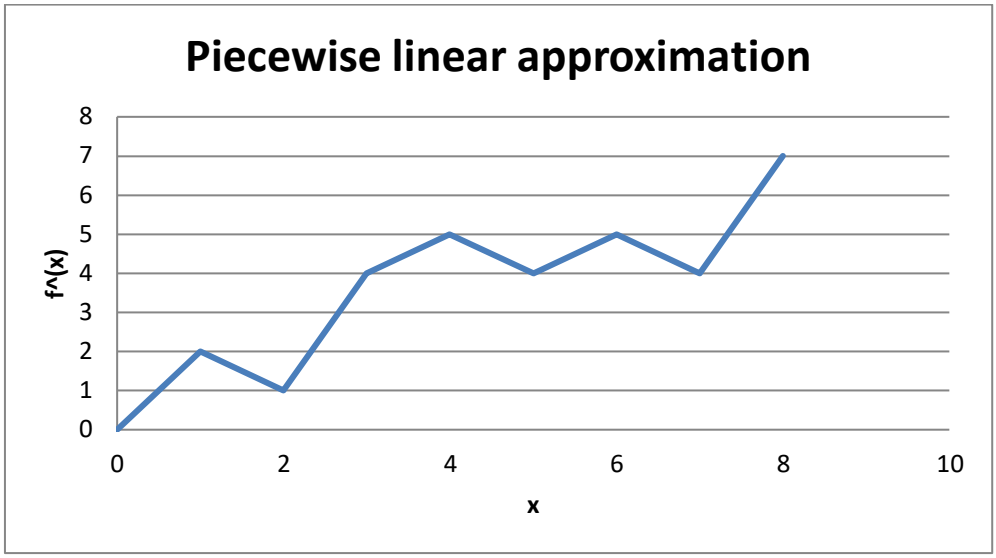
For example, the following graph is of $\chi_2(x)$.



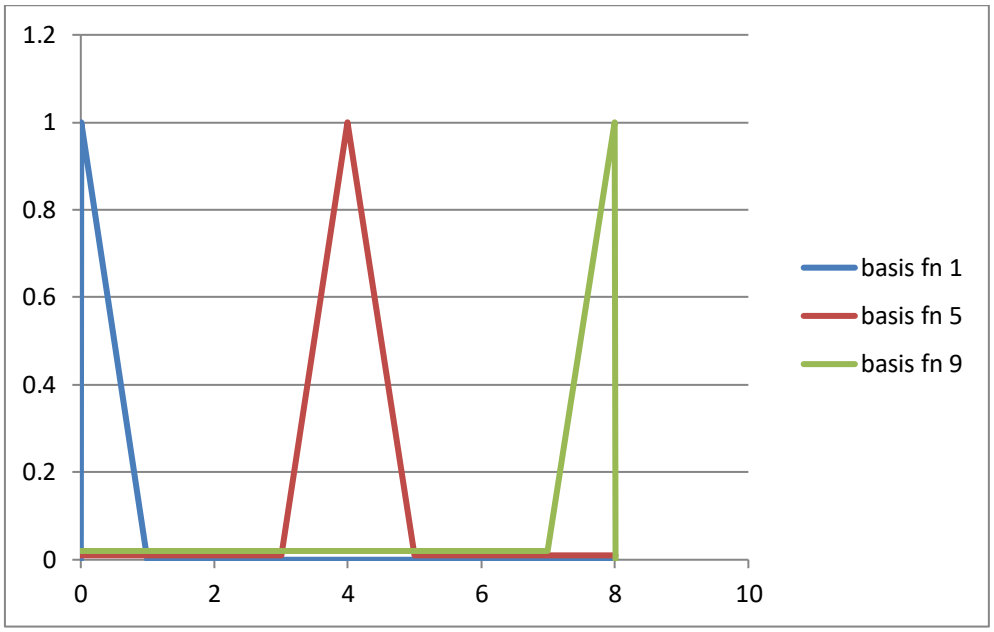
The nine basis functions all have the same shape.

Linear basis functions

The representation of the data using a piecewise linear approximation is shown in the following graph.

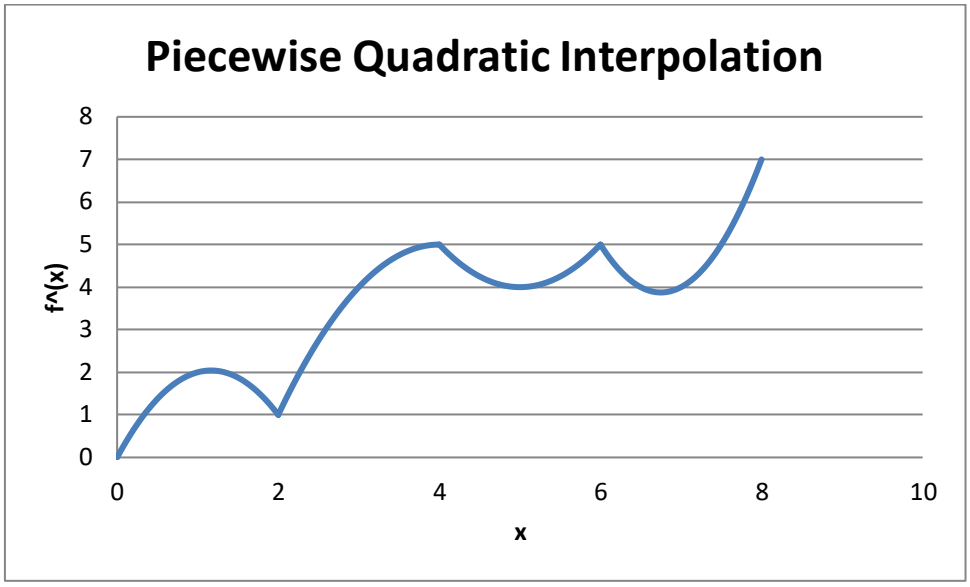


The linear basis functions are illustrated in the following in the following graph. Note the basis functions on the edges are different from those within the domain.



Quadratic basis functions

The representation of the data using a piecewise quadratic approximation is shown in the following graph.



The quadratic basis functions are illustrated in the following in the following graph. Again the basis functions on the edges are different from those within the domain.